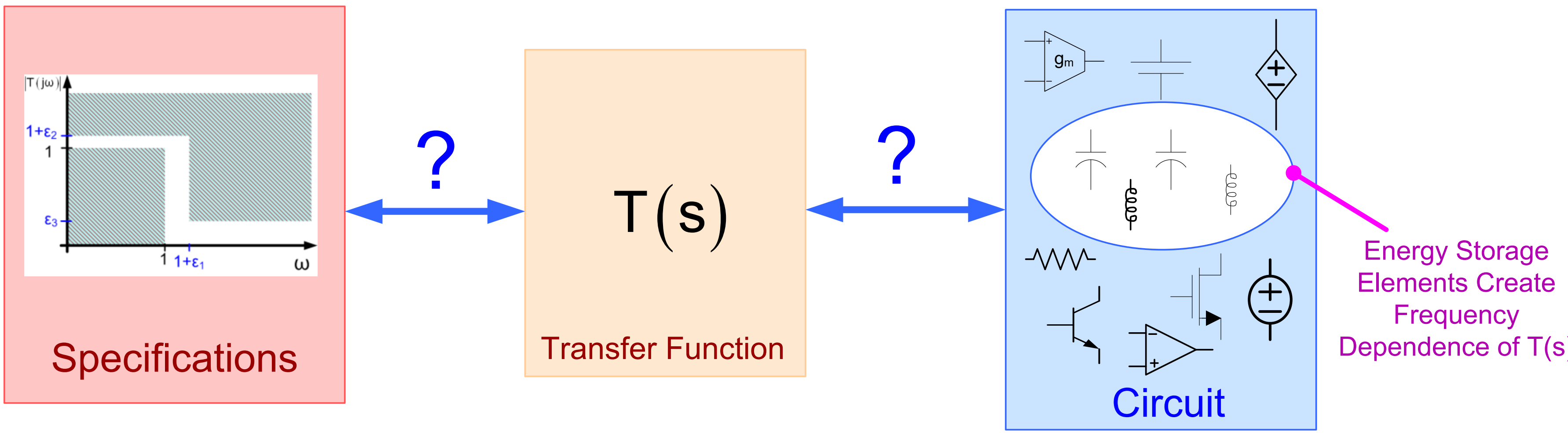
EE 508
Lecture 3

Filter Concepts/Terminology

Basic Properties of Electrical Circuits

Review from Last Time

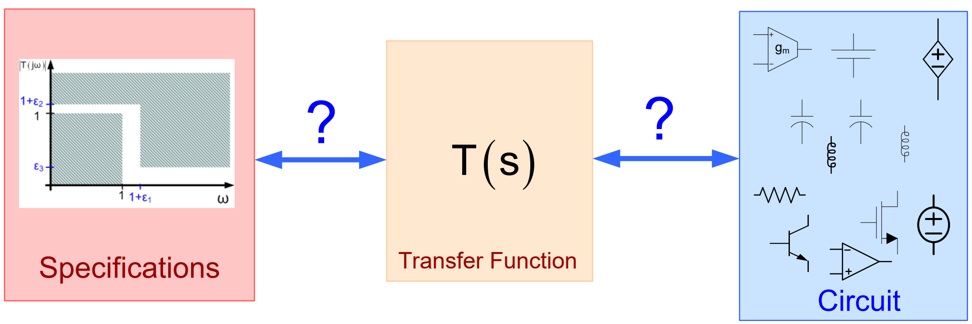
Is there a systematic way to design filters?



Filter Design Strategy: Use the transfer function as an intermediate step between the Specifications and Circuit Implementation

Review from Last Time

Filter Design Process



Filter Design Strategy: Use the transfer function as an intermediate step between the Specifications and Circuit Implementation

Establish Specifications
- possibly $T_D(s)$ or $H_D(z)$
- magnitude and phase characteristics or restrictions
- time domain requirements

↓

Approximation
- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
- possibly acceptable realizable time-domain responses

↓

Synthesis
- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

↓

Filter

Filter Design Process

Review from Last Time

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- obtain acceptable transfer functions $T_A(s)$ or $H_A(z)$
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Synthesis

- build circuit or implement algorithm that has response close to $T_A(s)$ or $H_A(z)$
- actually realize $T_R(s)$ or $H_R(z)$

↓

Filter

Must understand the real performance requirements

- **Many acceptable specifications for a given application**
- **Some much better than others**
- **But often difficult to obtain even one that is useful**

Obtain an acceptable approximating function

- **Many acceptable approximating functions for a given specification**
- **Some much better than others**
- **But often difficult to obtain even one!**

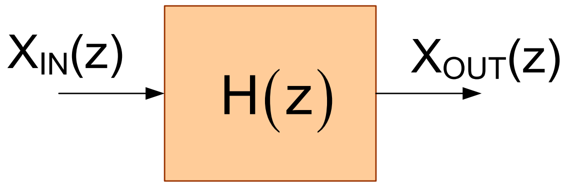
Design (synthesize) a practical circuit that has a transfer function close to the acceptable transfer function

- **Many acceptable circuits for a given approximating function**
  Some much better than others
- **But often difficult to obtain even one!**

Important to make good decisions at each step in the filter design process because poor decisions will not be absolved in subsequent steps

Review from Last Time

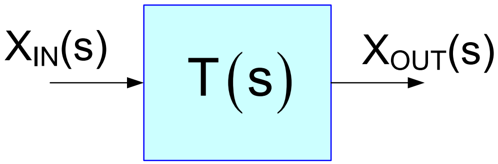
Filter Concepts and Terminology

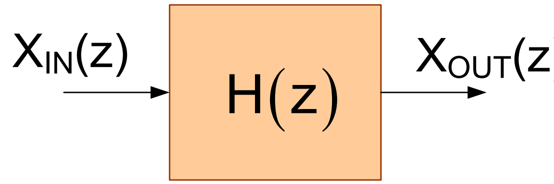


$$H(z) = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{n} b_i z^i} = \frac{N(z)}{D(z)}$$

- A polynomial is said to be "integer monic" if the coefficient of the highest-order term is 1
- If D(z) is integer monic, then N(z) and D(z) are unique
- If D(z) is integer monic, then the a_k and b_k terms are unique
- The roots of N(z) are termed the zeros of the transfer function
- The roots of D(z) are termed the poles of the transfer function
- If N(z) and D(z) are of orders m and n respectively, then there are m zeros and n poles in H(z)

Filter Concepts and Terminology





$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = \frac{N(s)}{D(s)}$$

$$H(z) = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{n} b_i z^i} = \frac{N(z)}{D(z)}$$

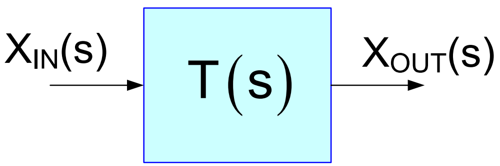
- Key Theorem: The continuous-time filter is stable iff all poles lie in the open left half of the s-plane

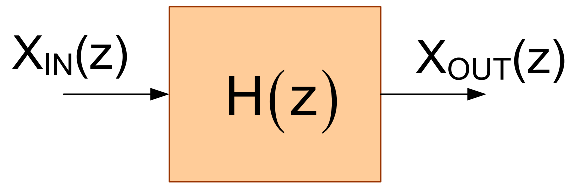
- Key Theorem: The discrete-time filter is stable iff all poles lie in the open unit circle

- The zeros of T(s) need not lie in the left half plane to maintain stability

- The zeros of H(z) need not lie in the open unit circle to maintain stability

Filter Concepts and Terminology



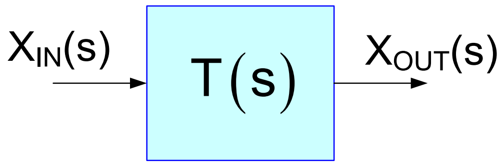


$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = \frac{N(s)}{D(s)}$$

$$H(z) = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{n} b_i z^i} = \frac{N(z)}{D(z)}$$

- Filter stability is of concern at the approximation stage of the filter design process
- Filter stability is required but not of concern at the synthesis stage for any useful filter (this concept is often misrepresented in the industry)
- Oscillators can be viewed at “unstable” filters
- An unstable filter will ultimately (and very quickly) either oscillate or latch up

Filter Concepts and Terminology



Two Approximations with Identical Magnitude Responses:

$$T_1(s) = \frac{1}{s+1} \qquad\qquad T_2(s) = \frac{1}{s-1}$$

$$\left|T_1(j\omega)\right| = \left|T_2(j\omega)\right| = \sqrt{\frac{1}{1+\omega^2}}$$

- Filter stability is of concern at the approximation stage of the filter design process
- $T_1(s)$ is a stable approximation $T_2(s)$ is an unstable approximation

Will a circuit that implements $T_2(s)$ oscillate?

Filter Concepts and Terminology

Minimum Phase Property

- An s-domain rational fraction is termed minimum-phase if all poles and all zeros have a non-positive real part
- An s-domain rational fraction is minimum-phase if it has no poles or zeros in the RHP or on the imaginary axis
- A z-domain rational fraction is minimum-phase if the magnitude of all poles and zeros are less that 1
- A z-domain rational fraction is minimum-phase iff no poles or zeros lie on or outside of the unit circle

Filter Concepts and Terminology

Measures

$$< \;\; ? \;\; \leq \qquad\qquad\qquad \geq \;\; ? \;\; >$$

How important is it to distinguish between these quantities when considering continuous-time filter concepts?

For example, consider a pole of a filter $P=\alpha+j\beta$

A filter is stable if the real part of the poles are in the left half-plane

$$\alpha < 0$$

A filter is stable if the real part of the poles are not in the right half-plane

$$\alpha \leq 0$$

It is of no concern to distinguish between these two conditions !!

Filter Concepts and Terminology

Measures

$<$? $\leq$ $\geq$? $>$

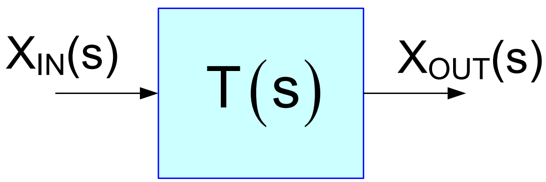
How important is it to distinguish between these quantities when considering continuous-time filter concepts?

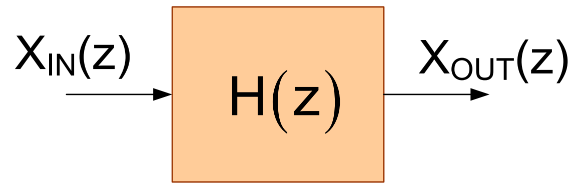
Any point or any line in the complex plane is of Euclidian measure 0

No continuous-time filter has even been built that has had a pole or zero on the imaginary axis for anything longer than infinitely small time

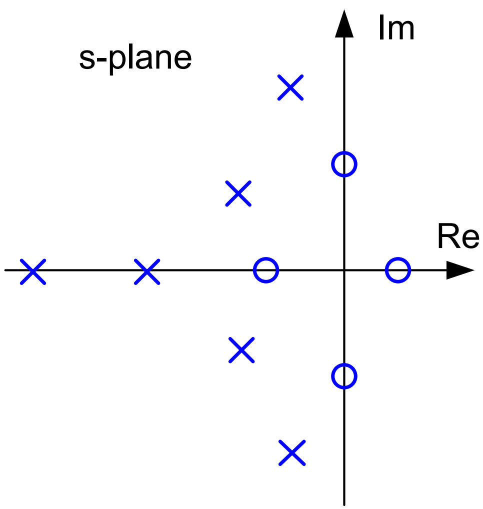
The probability is 0 that a filter can ever be built that has a pole or zero on any predetermined line or at any predetermined point in the complex plane (for longer than infinitely small time)

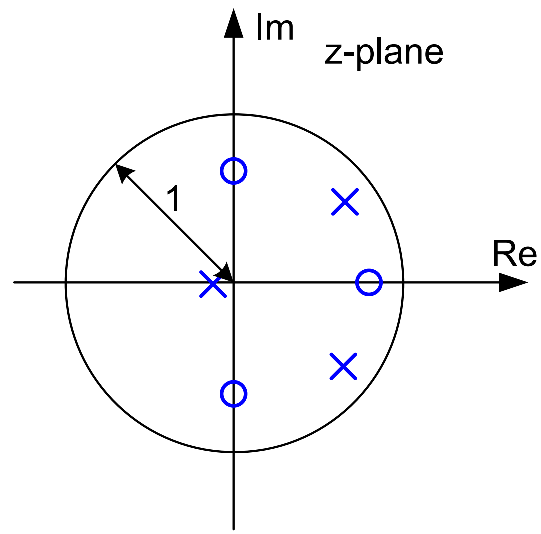
Filter Concepts and Terminology





Pole-zero Plots





Filter Concepts and Terminology

$$X_{IN}(s) \longrightarrow \boxed{T(s)} \longrightarrow X_{OUT}(s)$$

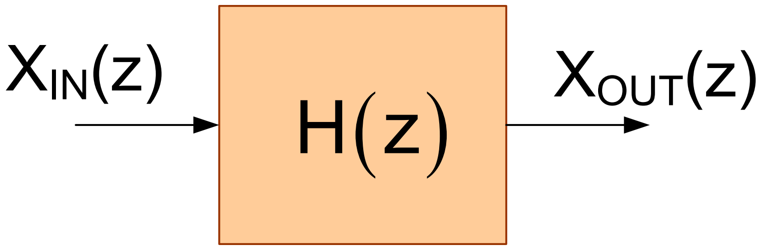
$$T(s) = \frac{\sum_{i=0}^{m} a_i s^i}{\sum_{i=0}^{n} b_i s^i} = \frac{N(s)}{D(s)}$$

- If T(s) is a rational fraction with poles and/or zeros in the RHP, then $\tilde{T}(s)$ obtained by reflecting all RHP roots around the imaginary axis back into the LHP has the following properties

a) minimum phase
b) stable
c) $\left|\tilde{T}(s)\right|_{s=j\omega} = \left|T(s)\right|_{s=j\omega}$ for all ω

Note the phase of T(s) and $\tilde{T}(s)$ will differ

Filter Concepts and Terminology



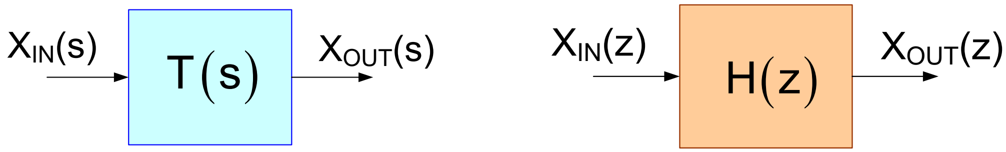
$$H(z) = \frac{\sum_{i=0}^{m} a_i z^i}{\sum_{i=0}^{n} b_i z^i} = \frac{N(z)}{D(z)}$$

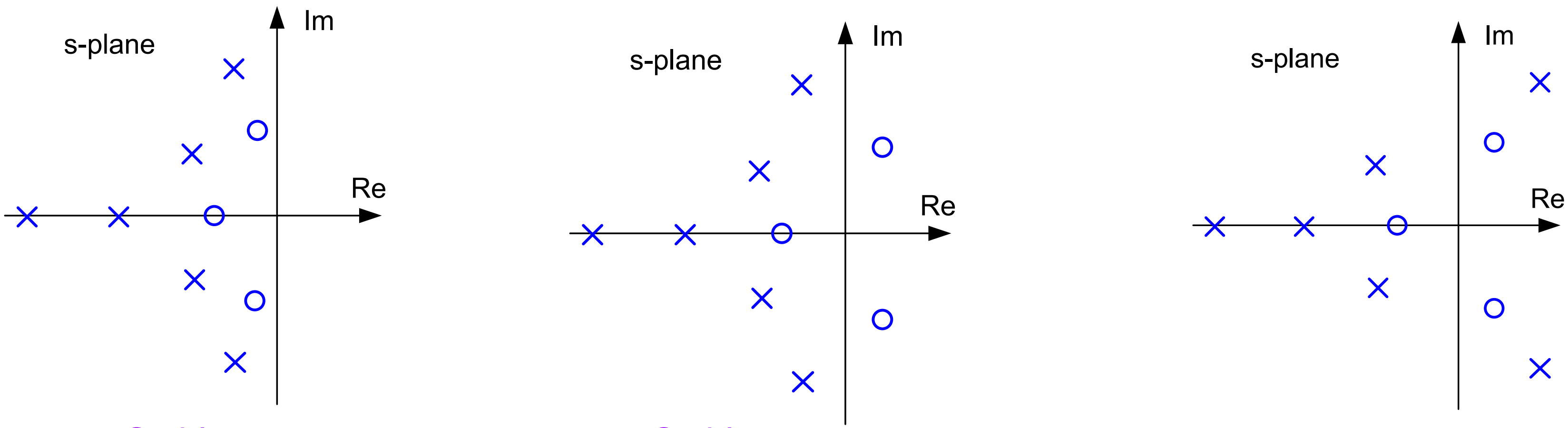
If H(z) is a rational fraction with poles and/or zeros outside the unit circle, then $\tilde{H}(z)$ obtained by reflecting all roots outside the unit circle back into the unit circle by the complex conguate reciprocal reflection and then scaling the transfer function by the magnitude of the reciprocal of the root has the following properties

a) minimum phase
b) stable
c) $\left|\tilde{H}(z)\right|_{z=e^{j\omega T}} = \left|H(z)\right|_{z=e^{j\omega T}}$ for all ω

Note the phase of H(z) and $\tilde{H}(z)$ will differ

Filter Concepts and Terminology

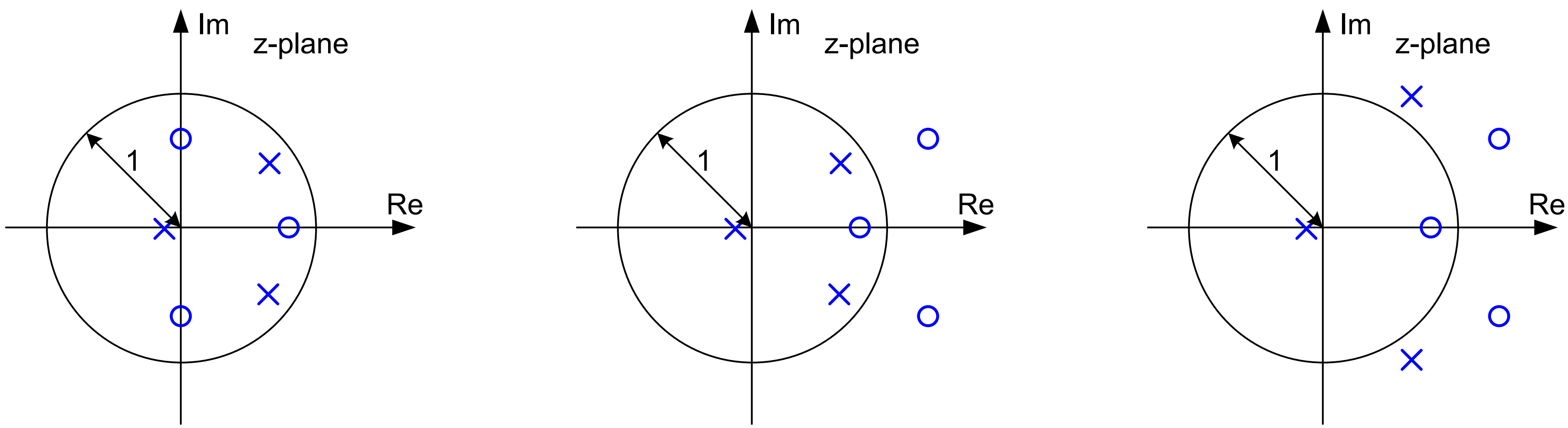




- Stable
- Minimum Phase

- Stable
- Not minimum Phase

- Not stable
- Not minimum Phase



Example: Non-minimum Phase Transfer Function

$$T(s) = \frac{s-1}{s+1}$$

$$\left|T(j\omega)\right| = \sqrt{\frac{\omega^2+(-1)^2}{\omega^2+1^2}}^{2\ 2} = 1$$

$$\angle T(j\omega) = \frac{\tan^{-1}\left(\frac{\omega}{-1}\right)}{\tan^{-1}\left(\frac{\omega}{1}\right)}$$

Though the magnitude of the gain is 1, non-minimum phase filters are often used for phase adjustment

Beware that the arctan function is multi-valued and in CAD tools gives “a” principle value that may or may not consider the quadrant of the two arguments

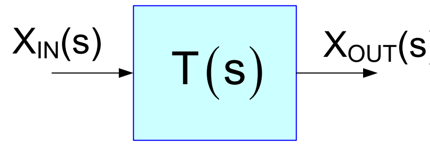
Example: Non-minimum Phase Transfer Function

$$T(s) = \frac{s^2 - as + 1}{s^2 + as + 1}$$

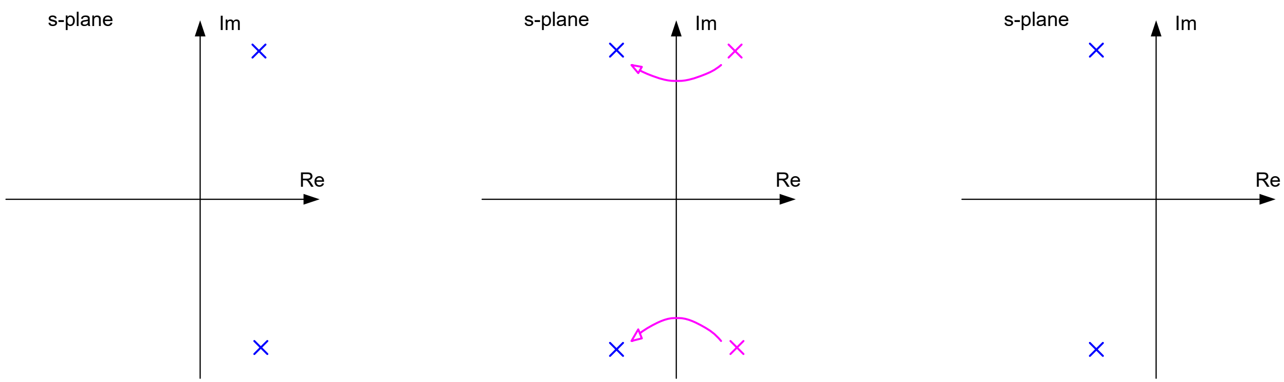
$$\left|T(j\omega)\right| = \sqrt{\frac{\left(1-\omega^2\right)^2 + a^2\omega^2}{\left(1-\omega^2\right)^2 + (-a)^2\,\omega^2}} = 1$$

$$\angle T(j\omega) = \frac{\tan^{-1}\left(\frac{-a\omega}{1-\omega^2}\right)}{\tan^{-1}\left(\frac{a\omega}{1-\omega^2}\right)}$$

Filter Concepts and Terminology



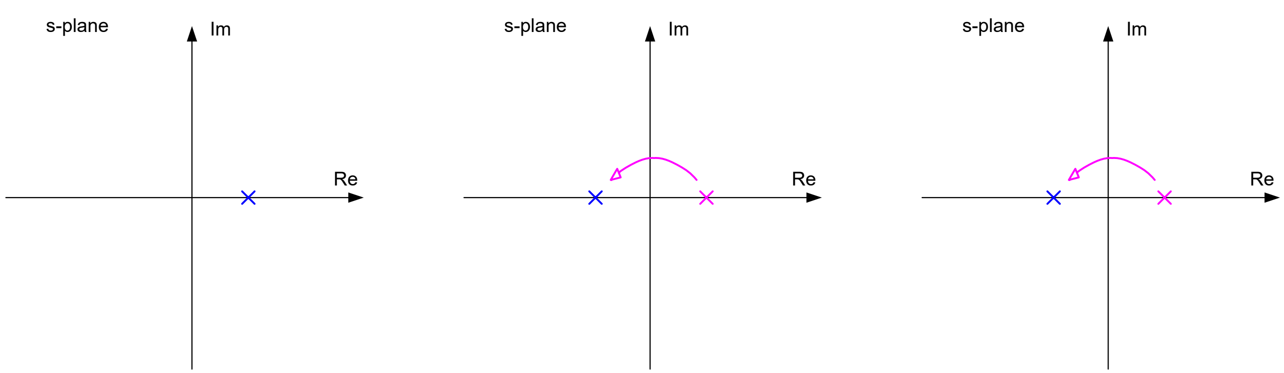
Reflecting poles and zeros to maintain stability or establish minimum phase



Not minimum Phase

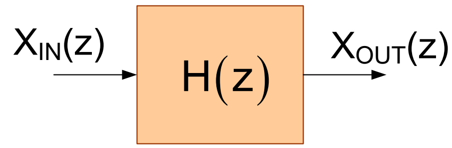
Reflection

Minimum Phase



Note: magnitude of real part is preserved in reflection, imaginary part remains unchanged

Filter Concepts and Terminology



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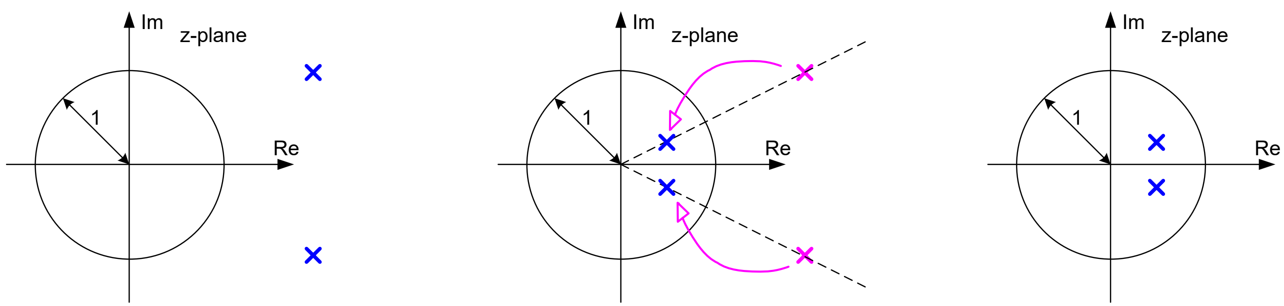
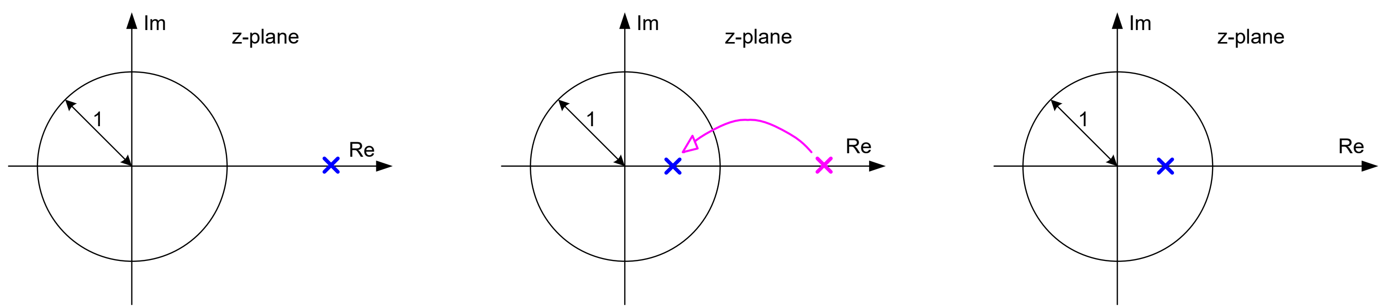
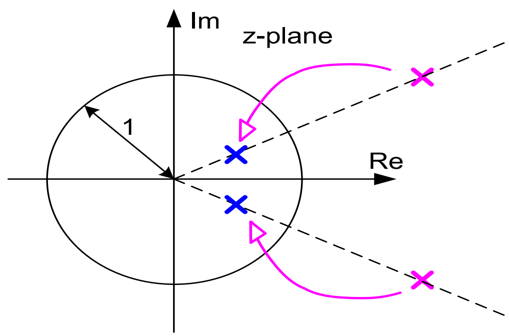


Not minimum Phase | Reflection | Minimum Phase



Note: complex conjugate reciprocal reflection maintains angle but magnitude of reflected root is the reciprocal of the magnitude of the original root

Complex Conjugate Reciprocal Reflection



Express X in polar form as

$$X = \mathrm{R}\, e^{j\theta}$$

The complex conjugate reciprocal reflection is

$$X_{CCRT} = \mathrm{R}^{-1}\, e^{j\theta}$$

Filter Concepts and Terminology

- 2-nd order polynomial characterization
- Biquadratic Factorization
- Op Amp Modeling
- Stability and Instability
- Roll-off characteristics
- Dead Networks
- Root Characterization
- Scaling, normalization, and transformation

2-nd order polynomial characterization

$$s^2+as+b \qquad \{a,b\}$$

$$s^2+\frac{\omega_o}{Q}s+\omega_0^2 \qquad \{\omega_o,Q\}$$

$$s^2+2\zeta\omega_0 s+\omega_0^2 \qquad \{\zeta,\ \omega_o\}$$

$$s^2+(p_1+p_2)s+p_1p_2 = (s+p_1)(s+p_2) \qquad \{p_1,p_2\}$$

with complex conjugate roots

$$s^2+2\alpha s+\alpha^2+\beta^2 = (s+\alpha+j\beta)(s+\alpha-j\beta) \qquad \{\alpha,\beta\}$$

$$s^2+2r\cos(\theta)s+r^2 = (s+re^{j\theta})(s+re^{-j\theta}) \qquad \{r,\theta\}$$

2-nd order polynomial characterization

$\{a,b\}$ $\{\omega_o,Q\}$ $\{\zeta, \omega_o\}$ $\{p_1,p_2\}$

$\{\alpha,\beta\}$ $\{r,\theta\}$

Alternate equivalent parameter sets

Widely used interchangeably

Easy mapping from one to another

Defined irrespective of whether polynomial appears in numerator or denominator of transfer function

If order is greater than 2, often multiple root pairing options so these parameter sets will not be unique for a given polynomial or transfer function

If cc roots exist, these will almost always be paired together (unique)

Filter Concepts and Terminology

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Biquadratic Factorization

$$T(s) = \frac{\sum_{i=1}^{m} a_i s^i}{\sum_{i=1}^{n} b_i s^i} = \frac{N(s)}{D(s)}$$

If m or n is even, integer-monic polynomials N(s) or D(s) can be expressed as

$$P(s) = \sum_{i=0}^{k} c_i s^i = \prod_{i=1}^{k/2} (s^2 + d_{1i} s + d_{2i})$$

If m or n is odd, integer-monic polynomials N(s) or D(s) can be expressed as

$$P(s) = \sum_{i=0}^{k} c_i s^i = (s + d_0) \prod_{i=1}^{(k-1)/2} (s^2 + d_{1i} s + d_{2i})$$

- These are termed quadratic factorizations

- If both N(s) and D(s) are expressed as quadratic factorizations, quadratic pairs can be grouped to obtain a Biquadratic factorization of T(s)

Biquadratic Factorization

Pole and zero pairings of realizable transfer functions can always be made so that all coefficients in the biquadratic factorizations are real

In general, the biquadratic factorizations are not unique

-If roots are real, multiple choices for first-order factor and remaining roots can be partitioned into groups of 2 in different ways

-Complex conjugate root pairs are generally grouped together so that all Coefficients are real

Biquadratic Factorization

If n is even and n≥m,

$$T(s)=\frac{\sum_{i=1}^{m} a_i s^i}{\sum_{i=1}^{n} b_i s^i}=K\bullet\prod_{i=1}^{n/2} T_{BQi}(s)$$

If n is odd and n≥m,

$$T(s)=\frac{\sum_{i=1}^{m} a_i s^i}{\sum_{i=1}^{n} b_i s^i}=K\bullet\left(\frac{a_{10}s+a_{00}}{s+b_{00}}\right)\bullet\prod_{i=1}^{(n-1)/2} T_{BQi}(s)$$

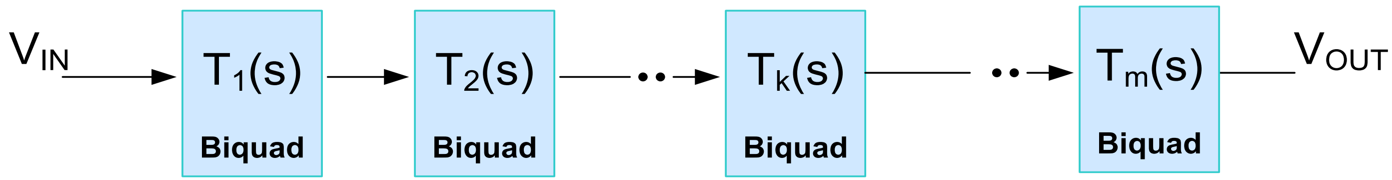
where $T_{BQi}(s)=\dfrac{a_{2i}s^2+a_{1i}s+a_{0i}}{s^2+b_{1i}s+b_{0i}}$

and where K is a real constant and all coefficients are real (some may be 0)

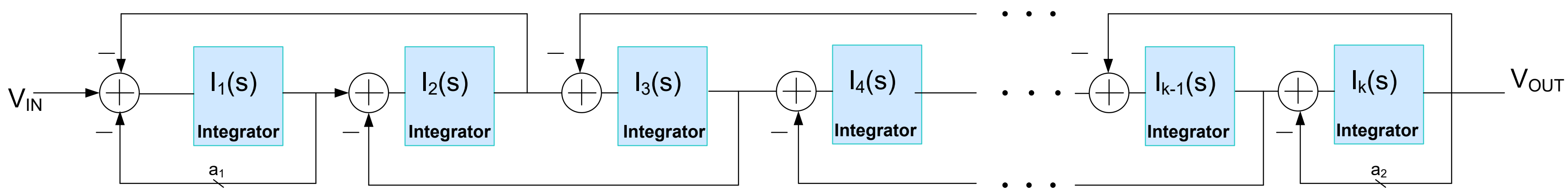
- Factorization is not unique
- H(z) factorizations not restricted to have m≤n
- Each biquatratic factor can be represented by any of the 6 alternative parameter sets in the numerator or denominator

Common Filter Architectures

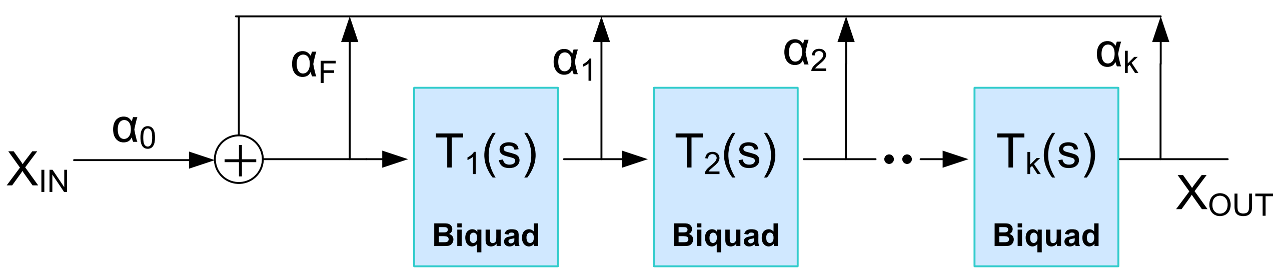
Cascaded Biquads



Leapfrog



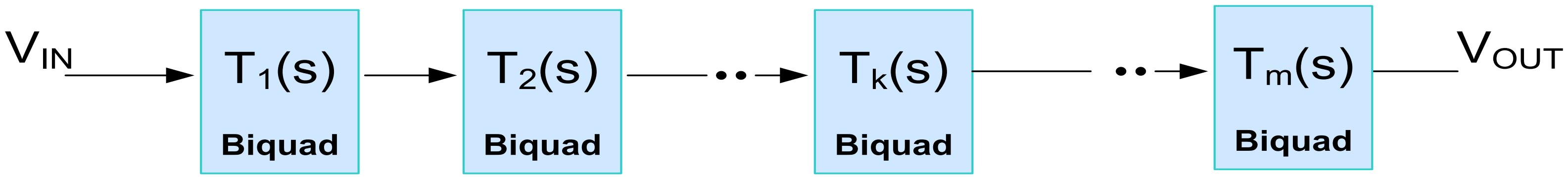
Multiple-loop Feedback



- Three classical filter architectures are shown
- The Cascaded Biquad and the Leapfrog approaches are most common
- The Cascaded Biquad structure follows directly from the Biquadratic Factorization

Common Filter Architectures

Cascaded Biquads



$$T(s) = T_1 T_2 \bullet\bullet T_m$$

- Sequence in Cascade does not affect the approximation
- Sequence in Cascade often affects performance of actual implementations
- Different biquadratic factorizations will provide different performance
- Although some attention was given to the different alternatives for biquadratic factorization, a solid general formulation of the cascade sequencing problem or the biquadratic factorization problem never evolved

Stay Safe and Stay Healthy !

End of Lecture 3